Paper

On Some Diophantine Equations (III)

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In this paper, we study about the equation $(2^{x_{12}}-1)/b^{y_1}=(b^{y_{12}}-1)/2^{x_1}=k$, where $b,\ k$ are odd. By considering the factorization into prime factors, we find solutions of the equation. In the case of $k=l_1^{s_1}$ and $y_{12}\neq 1$, if b is a prime number, then the equation has two solutions. And if $b=l_3^{s_3}l_4^{s_4}$ is satisfied, then the equation has no solutions. In the case of $k=l_1^{s_1}l_1^{s_2}$ and $x_{12}\equiv 0 \pmod 2$, if b is a prime number, then the equation has no solutions. In case of $k=l_1^{s_1}l_1^{s_2}$ and $x_{12}\equiv 0 \pmod 4$, if $b=l_3^{s_3}l_4^{s_4}$ is satisfied, then the equation has one solution.

Keywords : Diophantine equation, Existence of solutions, Factorization into prime factors, Catalan's Theorem

1. INTRODUCTION

Let $\mathbb{N}$ be a set of positive integers. In this paper, we use the following variables:

1) Let $a,\ b\in\mathbb{N}\setminus\{1\}$, and let $x,\ x_1,\ x_2,\ x_{12},\ y,\ y_1,\ y_2,\ y_{12}\in\mathbb{N}$,

2) Let $l_1,\ l_2,\ l_3,\ l_4$ be distinct odd prime numbers, and let $s_1,\ s_2,\ s_3,\ s_4\in\mathbb{N}$.

Let $c\in\mathbb{N}$. The equation $a^x-b^y=c$ has been studied by many authors. Especially, in the case of $c=1$, the following Catalan's Theorem[(1)] is well known:

Catalan's Theorem Let $x,\ y>1$. Then equation $a^x-b^y=1$ has an unique solution $3^2-2^3=1$.

M.A.Bennett[(2)] shows that the equation $a^x-b^y=c$ has at most two solutions for $(x,\ y)$. And we know the following eleven cases:

$$3^1-2^1=3^2-2^3=1,\quad 2^3-3^1=2^5-3^3=5,\quad 2^4-3^1=2^8-3^5=13,\quad 2^3-5^1=2^7-5^3=3,$$

$$13^1-3^1=13^3-3^7=10,\quad 91^1-2^1=91^2-2^{13}=89,\quad 6^1-2^1=6^2-2^5=4,\quad 15^1-6^1=15^2-6^3=9, \tag{1.1}$$

$$280^1-5^1=280^2-5^7=275,\quad 4930^1-30^1=4930^2-30^5=4900,\quad 6^4-3^4=6^5-3^8=1215.$$

Furthermore, M.A. Bennett refers to the following conjecture:

Conjecture The equation $a^x-b^y=c$ has at most one solution for $(x,\ y)$ except eleven cases of (1.1).

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Let $\gcd(a,\ b)=1$, and let $x_1<x_2,\ y_1<y_2$. Then the equation $a^{x_1}-b^{y_1}=a^{x_2}-b^{y_2}=c$ is transformed into the equation

$$(1.2)\qquad \frac{a^{x_{12}}-1}{b^{y_1}}=\frac{b^{y_{12}}-1}{a^{x_1}}=k\ ,$$

where $x_{12}=x_2-x_1,\ y_{12}=y_2-y_1$ and k is a suitable positive integer. N.Kobachi, Y.Motoda and Y.Yamahata[(3)] have so far studied the equation (1.2), where $a,\ b$ are distinct prime numbers, on the cases of $k=1,\ 2,\ 3,\ 4,\ 5$ and prime numbers with $k\ge 7$. Furthermore, in this paper, we study the equation (1.2), where $a=2$, on the cases $k=l_1^{s_1}$ and $k=l_1^{s_1}l_2^{s_2}$

2. FACTORIZATION INTO PRIME FACTORS

In this section, we prepare some lemmas.

Lemma 2.1 Suppose $a\ge 3$ and that x is odd with $x\ne 1$.

(1) There exists at least one odd prime number p such that $p\mid(a^x-1)/(a-1)$ and $p\nmid a-1$ are satisfied.

(2) There exists at least one odd prime number p such that $p\mid(a^x+1)/(a+1)$ and $p\nmid a+1$ are satisfied.

Proof We show the proof of (1) only. The proof of (2) is shown similarly in the case of (1).

We remark that $a-1<(a^x-1)/(a-1)$ and $(a^x-1)/(a-1)$ is odd.

Now, put $K=\gcd\{(a^x-1)/(a-1),\ a-1\}$. If $K=1$ is satisfied then the result of (1) follows. After this, we suppose K is odd with $K\ge 3$. From $K\mid a-1$, $a\equiv 1 \pmod K$ is obtained. Thus $(a^x-1)/(a-1)\equiv x \pmod K$ follows. Furthermore, from $K\mid(a^x-1)/(a-1)$, $K\mid x$ is satisfied. This leads $a^K-1\mid a^x-1$.

And, from $a\equiv 1 \pmod K$, there is a positive integer A such that $a=KA+1$. Therefore $a^K-1=\sum_{j=1}^{K}{}_KC_j(KA)^j$, and so

$$(2.1)\qquad \frac{1}{K}\cdot\frac{a^K-1}{a-1}=1+KA\left\{\frac{K-1}{2}+A\sum_{j=3}^{K}{}_KC_j(KA)^{j-3}\right\}$$

follows. Let $m\in\mathbb{N}$ and let $t_i\in\mathbb{N}$. If K is factorized by m distinct odd prime numbers p_j, that is $K=\prod_{j=1}^{m}p_j^{t_j}$, then we have $(a^K-1)/K(a-1)\equiv 1 \pmod{p_j}$ for $j=1,\ \cdots,\ m$. Thus, from $a^K-1\mid a^x-1$, the result of (1) follows. □

Lemma 2.2 Suppose $a\ge 3$ and that x is odd with $x\ne 1$.

(1) If $a-1$ is factorized by m distinct prime numbers, then a^x-1 contains at least $m+1$ distinct prime numbers as the factor.

(2) If $a+1$ is factorized by m distinct prime numbers, then a^x+1 contains at least $m+1$ distinct prime numbers as the factor.

(3) If a^2-1 is factorized by m distinct prime numbers, then $a^{2x}-1$ contains at least $m+2$ distinct prime numbers as the factor.

Proof From Lemma 2.1, the results (1) and (2) are clear. We show the proof of (3) only.

From (1) of Lemma 2.1, there is at least one odd prime p such that $p\mid(a^x-1)/(a-1)$ and $p\nmid a-1$ are satisfied. If $p\mid a^x+1$, then we have a contradiction to $\gcd(a^x+1,\ a^x-1)=\gcd(2,\ a^x-1)=1$ or 2. Thus $p\nmid a^x+1$ follows. Similarly, from (2) of Lemma 2.1, there is at least an odd prime q such that $q \mid (a^x+1)/(a+1)$ and $q \nmid a+1$ are satisfied. Furthermore $q \nmid a^x-1$ follows. That is $p \neq q$ and $p,\ q \nmid a^2-1$. Thus, from the equation

(2.2)
$$a^{2x}-1=(a^x-1)(a^x+1)=\left\{(a-1)\left(\frac{a^x-1}{a-1}\right)\right\}\left\{(a+1)\left(\frac{a^x+1}{a+1}\right)\right\}=(a^2-1)\left(\frac{a^x-1}{a-1}\right)\left(\frac{a^x+1}{a+1}\right),$$

we can obtain the result (3). □

Lemma 2.3 If $b=2^{4m+1}+1$ is satisfied, then the equation $b^2+1=2^x\cdot l_1^{s_1}$ has no solutions.

Proof Put $A=2^{4m}$. Then we have

(2.3)
$$2^x\cdot l_1^{s_1}=(2A+1)^2+1=2(2A^2+2A+1).$$

Thus $x=1$ and $l_1^{s_1}=2A^2+2A+1$ is obtained. And, from $A=16^m\equiv 1 \pmod 5$, $l_1^{s_1}\equiv 0 \pmod 5$ follows. That is $l_1=5$. Therefore $5^{s_1}=2A^2+2A+1$ is satisfied. Then we remark that $A\geq 16$ leads $s_1\geq 3$. Then we have

(2.4)
$$5^2(5^{s_1-2}-1)=2A^2+2A-24=2(A+4)(A-3)=2^3(2^{4m-2}+1)(2^{4m}-3).$$

Thus $8 \mid 5^{s_1-2}-1$ and so s_1-2 is even. That is $3 \mid 5^{s_1-2}-1$. Therefore $2A^2+2A-24\equiv 0 \pmod 3$ follows. On the other hand, from $A\equiv 1 \pmod 3$, $2A^2+2A-24\equiv 1 \pmod 3$ follows. Thus we have a contradiction. □

Lemma 2.4 We have the following result:

(1) If x is odd with $x\geq 5$, then 2^x+1 has an odd prime number p other than 3 and 3 as the factor.

(2) If x is even with $x\geq 4$, then 2^x-1 has an odd prime number p other than 3 and 3 as the factor.

Proof (1) Since x is odd, $3 \mid 2^x+1$ follows. If $2^x+1=3^y$ is satisfied, from Catalan's Theorem, then there are only two solutions $(x,\ y)=(1,\ 1),\ (3,\ 2)$. Thus we have a contradiction to $x\geq 5$.

(2) Since x is even, $3 \mid 2^x-1$ follows. If $2^x-1=3^y$ is satisfied, from Catalan's Theorem, then there is an unique solution $(x,\ y)=(2,\ 1)$. Thus we have a contradiction to $x\geq 4$. □

Lemma 2.5 Let $x_1<x_2$. Then $x_2\equiv 0 \pmod 2$ and $x_1 \mid x_2/2$ follow if and only if $2^{x_1}+1 \mid 2^{x_2}-1$ is satisfied.

Proof Let R be the remainder of $(2^{x_2}-1)/(2^{x_1}+1)$, and let m be the quotient of x_2/x_1. Then the relation $R=(-1)^m\cdot 2^{x_2-mx_1}-1$ follows. Therefore $m\equiv 0 \pmod 2$ and $x_2=mx_1$ if and only if $R=0$ is satisfied. Thus the proof is completed. □

Remark 2.6 Similarly, we have the following results:

(1) Let $x_1\leq x_2$. Then $x_1 \mid x_2$ follows if and only if $2^{x_1}-1 \mid 2^{x_2}-1$ is satisfied.

(2) Let $x_1\leq x_2$. Then $x_2\equiv 1 \pmod 2$ and $x_1 \mid x_2$ follows if and only if $2^{x_1}+1 \mid 2^{x_2}+1$ is satisfied.

(3) Let $x_1\leq x_2$. Then $x_2\equiv 1 \pmod 2$ and $x_1=2$ follows if and only if $2^{x_1}-1 \mid 2^{x_2}+1$ is satisfied.

3. FACTORIZATION ON $2^{2x}-1$

In this section, we consider the factorization on $2^{2x}-1$. And we treat the following three cases:

$$2^{2x}-1=l_1^{s_1}\cdot l_2^{s_2}, \qquad 2^{2x}-1=l_1^{s_1}\cdot l_2^{s_2}\cdot l_3^{s_3}, \qquad 2^{2x}-1=l_1^{s_1}\cdot l_2^{s_2}\cdot l_3^{s_3}\cdot l_4^{s_4}.$$

Lemma 3.1 Let $k\in\mathbb{N}$. The system of equations $\begin{cases}2^x-1=1\\2^x+1=k\end{cases}$ has no solutions except $k=3$.

Proof From $k=(2^x-1)+2=3$, it is clear. □

Lemma 3.2 Let $k\in\mathbb{N}$. The system of equations $\begin{cases}2^x-1=3^y\\2^x+1=k\end{cases}$ has no solutions except $k=5$.

Proof From the Catalan's Theorem, the equation $2^x-1=3^y$ has an unique solution $(x,\ y)=(2,\ 1)$. Thus $k=2^2+1=5$ follows. The proof is completed. □

Lemma 3.3 Let $k\in\mathbb{N}$. The system of equations $\begin{cases}2^x-1=k\\2^x+1=3^y\end{cases}$ has no solutions except $k=1,\ 7$.

Proof From Catalan's Theorem, the equation $2^x+1=3^y$ has only two solutions $(x,\ y)=(1,\ 1),\ (3,\ 2)$. Thus, as $x=1,\ 3$, $k=1,\ 7$ follows respectively. The proof is completed. □

Proposition 3. 4 The equation $2^{2x}-1=l_1^{s_1}\cdot l_2^{s_2}$ has two solutions

$$2^4-1=3^1\cdot 5^1, \qquad 2^6-1=3^2\cdot 7^1.$$

Proof From Lemma 3.1, we have $1<2^x-1<2^x+1<l_1^{s_1}\cdot l_2^{s_2}$. Furthermore, from $\gcd(2^x-1,\ 2^x+1)=1$, we may suppose that both $2^x-1=l_1^{s_1}$ and $2^x+1=l_2^{s_2}$ are satisfied. Then, from $3\,|\,2^{2x}-1$, Either $l_1=3$ or $l_2=3$ follows. Thus, from Lemma 3.2 and Lemma 3.3, The result is obtained. □

Lemma 3.5 Let $l_1,\ l_2>3$. If $2^{2x}-1=3^y\cdot l_1^{s_1}\cdot l_2^{s_2}$, then either of the following system of equations is satisfied:

$$\begin{cases}2^x-1=3^y\cdot l_1^{s_1}\\2^x+1=l_2^{s_2}\end{cases}, \qquad \begin{cases}2^x+1=3^y\cdot l_1^{s_1}\\2^x-1=l_2^{s_2}\end{cases}.$$

Proof From Lemma 3.1, we have $1<2^x-1<2^x+1<3^y\cdot l_1^{s_1}\cdot l_2^{s_2}$. Furthermore, from Lemma 3.2 and Lemma 3.3, the following two systems of equations are not satisfied:

$$\begin{cases}2^x-1=3^y\\2^x+1=l_1^{s_1}\cdot l_2^{s_2}\end{cases}, \qquad \begin{cases}2^x+1=3^y\\2^x-1=l_1^{s_1}\cdot l_2^{s_2}\end{cases}.$$

Thus the proof is completed. □

Lemma 3.6 Let $k\in\mathbb{N}$. The system of equations $\begin{cases}2^x-1=3^y\cdot l_1^{s_1}\\2^x+1=k\end{cases}$ has no solutions except $k=17,\ 65$.

Proof From $3\,|\,2^x-1$, x is even. Therefore, from Proposition 3.4, there are only two solutions $2^4-1=3^1\cdot 5^1$ and

$2^6 - 1 = 3^2 \cdot 7^1$ on the equation $2^x - 1 = 3^y \cdot l_1^{s_1}$. For each solutions, $k = 17,\ 65$ follows. □

Lemma 3.7 Let $k \in \mathbb{N} \setminus \{1\}$ and $l_2 > 3$. If the system of equations $\begin{cases} 2^x - 1 = l_2^{s_2} \\ 2^x + 1 = 3^y \cdot k \end{cases}$ has solutions, then the following results are satisfied: (1) $y = 1,\ s_2 = 1$, (2) l_2 is a Mersenne prime number with $l_2 \geq 31$.

Proof From $3 \mid 2^x + 1$, x is odd. Furthermore, when $x = 1,\ 3$ is satisfied, $3^y \cdot k = 3,\ 9$ follows respectively. Thus we have a contradiction. After this, we suppose that x is odd with $x \geq 5$. Then, from Catalan's Theorem, both $s_2 = 1$ and $l_2 = 2^x - 1$ follow. Thus l_2 is a Mersenne prime number with $l_2 \geq 31$. Furthermore, since x is at least an odd prime number with $x \geq 5$, $y = \nu_3(2^x + 1) = \nu_3(3) + \nu_3(x) = 1 + 0 = 1$ follows, where notation $\nu_p(\cdot)$ is s p-adic valuation. □

We consider solutions of the equation $2^{2x} - 1 = l_1^{s_1} \cdot l_2^{s_2} \cdot l_3^{s_3}$

Proposition 3.8 We have the following results:

(1) Suppose that x is even. Then there is an unique solution $2^8 - 1 = 3^1 \cdot 5^1 \cdot 17^1$.

(2) Suppose that x is odd. Let M be a Mersenne prime with $M \geq 31$. If there exists an odd prime p and a positive integer t such that $p^t = (M+2)/3$ follows, there is a solution $2^{2m} - 1 = 3^1 \cdot M^1 \cdot p^t$, where $m = \nu_2(M+1)$, for M.

Proof From $3 \mid 2^{2x} - 1$, we may put $l_3 = 3$ and $s_3 = y$. Thus, from Lemma 3.5, either of the following system of equations is satisfied:

$$\begin{cases} 2^x - 1 = 3^y \cdot l_1^{s_1} \\ 2^x + 1 = l_2^{s_2} \end{cases}, \qquad \begin{cases} 2^x + 1 = 3^y \cdot l_1^{s_1} \\ 2^x - 1 = l_2^{s_2} \end{cases}.$$

(1) Suppose that x is even. Then we have $\begin{cases} 2^x - 1 = 3^y \cdot l_1^{s_1} \\ 2^x + 1 = l_2^{s_2} \end{cases}$. Thus, from Lemma 3.6, the result is obtained.

(2) Suppose that x is odd. Then we have $\begin{cases} 2^x + 1 = 3^y \cdot l_1^{s_1} \\ 2^x - 1 = l_2^{s_2} \end{cases}$. Thus, from Lemma 3.7, the result is obtained. □

Proposition 3.9 The equation $2^{4x} - 1 = l_1^{s_1} \cdot l_2^{s_2} \cdot l_3^{s_3} \cdot l_4^{s_4}$ has two solutions

$$2^{12} - 1 = 3^1 \cdot 5^1 \cdot 7^1 \cdot 13^1, \qquad 2^{16} - 1 = 3^1 \cdot 5^1 \cdot 17^1 \cdot 257^1.$$

Proof From Lemma 3.1 and $\gcd(2^{2x} - 1,\ 2^{2x} + 1) = 1$, we have $1 < 2^{2x} - 1 < 2^{2x} + 1 < l_1^{s_1} \cdot l_2^{s_2} \cdot l_3^{s_3} \cdot l_4^{s_4}$. And, from $3 \mid 2^{2x} - 1$, we may put $l_4 = 3$ and $s_4 = y$. Furthermore, from Lemma 3.2, either of the following system of equations is satisfied:

$$\begin{cases} 2^{2x} - 1 = 3^y \cdot l_1^{s_1} \\ 2^{2x} + 1 = l_2^{s_2} \cdot l_3^{s_3} \end{cases}, \qquad \begin{cases} 2^{2x} - 1 = 3^y \cdot l_1^{s_1} \cdot l_2^{s_2} \\ 2^{2x} + 1 = l_3^{s_3} \end{cases}.$$

Suppose $\begin{cases} 2^{2x} - 1 = 3^y \cdot l_1^{s_1} \\ 2^{2x} + 1 = l_2^{s_2} \cdot l_3^{s_3} \end{cases}$. From Lemma 3.6, There is an unique solution $\begin{cases} 2^6 - 1 = 3^3 \cdot 7^1 \\ 2^6 + 1 = 5^1 \cdot 13^1 \end{cases}$.

Suppose $\begin{cases} 2^{2x} - 1 = 3^y \cdot l_1^{s_1} \cdot l_2^{s_2} \\ 2^{2x} + 1 = l_3^{s_3} \end{cases}$. If x is even, form Proposition 3.8, There is an unique solution $\begin{cases} 2^8 - 1 = 3^1 \cdot 5^1 \cdot 17^1 \\ 2^8 + 1 = 257^1 \end{cases}$. If x is odd, then $5 \mid 2^x + 1$ and so $l_3 = 5$ follows. Furthermore, from Catalan's Theorem, the equation $2^{2x} + 1 = 5^{s_3}$ has an unique solution $(x,\ s_3) = (1,\ 1)$. Then $3^y \cdot l_1^{s_1} \cdot l_2^{s_2} = 2^2 - 1 = 3$ follows. Thus we have a contradiction. □

4. FACTORIZATION ON $b^{2y}-1$

Let b be odd. In this section, we consider the factorization on b^y-1. And we treat the following two cases:

$$b^{2x}-1=2^x\cdot l_1^{s_1}, \qquad b^{2y}-1=2^x\cdot l_1^{s_1}\cdot l_2^{s_2}.$$

Lemma 4.1 Let $k\in\mathbb{N}\setminus\{1\}$ and $x\geq 3$. The system of equations $\begin{cases} b^y-1=2 \\ b^y+1=2^{x-1}\cdot k \end{cases}$ has no solutions.

Proof From $2^{x-1}\cdot k=(b^i-1)+2=4$, it is clear. □

We consider solutions of the equation $b^{2y}-1=2^x\cdot l_1^{s_1}$, where y is even.

Proposition 4.2 We have the following results:

(1) Suppose that b is a prime number. Then there are only four solutions

$$5^2-1=2^3\cdot 3^1, \qquad 3^4-1=2^4\cdot 5^1, \qquad 17^2-1=2^5\cdot 3^2, \qquad 7^2-1=2^4\cdot 3^1.$$

(2) Suppose that b contains at least two distinct odd prime numbers. Then, if l_1 is either a Fermat prime number F with $F\geq 17$ or a Mersenne prime number M_0 with $M_0\geq 7$, There is only one solution $b^2-1=2^{r+2}\cdot l_1^1$, where $b=2l_1-1,\ r=\nu_2(l_1-1)$ as $l_1=F$ or $b=2l_1+1,\ r=\nu_2(l_1+1)$ as $l_1=M_0$.

Proof We remark that $8\mid b^2-1$ leads $x\geq 3$. From Lemma 4.1 and $\gcd(b^y-1,\ b^y+1)=2$, we have $2<b^y-1<b^y+1<2^x\cdot l_1^{s_1}$. Therefore, either of the following two systems of equations follows:

$$\text{(4.1)} \qquad \begin{cases} b^y-1=2^{x-1} \\ b^y+1=2l_1^{s_1} \end{cases},$$

$$\text{(4.2)} \qquad \begin{cases} b^y+1=2^{x-1} \\ b^y-1=2l_1^{s_1} \end{cases}$$

First, we consider the case of (4.1).

If $x=3$ is satisfied, then $b^y=2^2+1=5$ and so $b=5,\ y=1$ are obtained. Furthermore, from $l_1^{s_1}=(5^1+1)/2=3$, we have $l_1=3$ and $s_1=1$. Thus $5^2-1=2^3\cdot 3^1$ follows.

If $x=4$ is satisfied, then $b^y=2^3+1=9$ and so $b=3,\ y=2$ are obtained. Furthermore, from $l_1^{s_1}=(3^2+1)/2=5$, we have $l_1=5$ and $s_1=1$. Thus $3^4-1=2^4\cdot 5^1$ follows.

If $x=5$ is satisfied, then $b^y=2^4+1=17$ and so $b=17,\ y=1$ are obtained. Furthermore, from $l_1^{s_1}=(17^1+1)/2=9$, we have $l_1=3$ and $s_1=2$. Thus $17^4-1=2^5\cdot 3^2$ follows.

Suppose $x\geq 6$. From Catalan's Theorem, $y=1,\ b=2^{x-1}+1$ are obtained. Furthermore we have $l_1^{s_1}=2^{x-2}+1$. And, from Catalan's Theorem, $s_1=1,\ l_1=2^{x-2}+1$ are obtained. Thus l_1 is a Fermat prime number F with $F\geq 17$. We remark that x is even at least. Then we have $b=2^{x-1}+1=2F-1$. And, from Lemma 2.4, b has an odd prime number p other than 3 and 3 as the factor.

Next, we consider the case of (4.2).

If $x=3$, then $b^y=2^2-1=3$ and so $b=3,\ y=1$ are obtained. Hence $l_1^{s_1}=(3^1-1)/2=1$ follows. Thus we have a contradiction.

If $x=4$, then $b^y=2^3-1=7$ and so $b=7,\ y=1$ are obtained. Furthermore, from $l_1^{s_1}=(7^1-1)/2=3$, we have $l_1=3$ and $s_1=1$. Thus $7^2-1=2^4\cdot 3^1$ follows.

Suppose $x\ge 5$. From Catalan's Theorem, $y=1,\ b=2^{x-1}-1$ are obtained. Furthermore we have $l_1^{s_1}=2^{x-2}-1$. And, from Catalan's Theorem, $s_1=1,\ l_1=2^{x-2}-1$ are obtained. Thus l_1 is a Mersenne prime number M_0 with $M_0\ge 7$. We remark that x is odd at least. Then we have $b=2^{x-1}-1=2M_0+1$. And, from Lemma 2.4, b has an odd prime number p other than 3 and 3 as the factor. □

Lemma 4.3 Suppose that y is even. The system of equations $\begin{cases} b^y-1=2^{x-1} \\ b^y+1=2\cdot l_1^{s_1}\cdot l_2^{s_2} \end{cases}$ has no solutions.

Proof Since y is even, we have $y\ge 2$ and $x\ge 4$. From Catalan's Theorem, The equation $b^y-1=2^{x-1}$ has an unique solution $3^2-1=2^3$. Thus $b=3,\ y=2$ and $x=4$ follow. Therefore $2\cdot l_1^{s_1}\cdot l_2^{s_2}=3^2+1=10$ is obtained. Thus we have a contradiction. □

We consider solutions of equation $b^{2y}-1=2^x\cdot l_1^{s_1}\cdot l_2^{s_2}$, where y is even.

Proposition 4.4 We have the following results:

(1) Suppose that b is a prime number. Then there are only three solutions

$5^4-1=2^4\cdot 3^1\cdot 13^1$, $3^8-1=2^5\cdot 5^1\cdot 41^1$, $7^4-1=2^5\cdot 3^1\cdot 5^2$.

(2) Suppose that b contains at least two distinct odd prime numbers. Then, let M_0 be a Mersenne prime with $M_0\ge 7$. And, put $b=2M_0+1,\ r=\nu_2(M_0+1)$. If there exist an odd prime number p and a positive integer t such that $p^t=(b^2+1)/2$, There is only one solution $b^4-1=2^{r+3}\cdot M_0^{\,1}\cdot p^t$.

Proof We remark that $8\mid b^2-1$ leads $x\ge 3$. From Lemma 4.1 and $\gcd(b^y-1,\ b^y+1)=2$, we have $2<b^y-1<b^y+1<2^{x-1}\cdot l_1^{s_1}$. And, Since $y\equiv 0 \pmod 2$ leads $8\mid b^y-1$, The system of equations $\begin{cases} b^y-1=2\cdot l_1^{s_1} \\ b^y+1=2^{x-1}\cdot l_2^{s_2} \end{cases}$ dose not occur. Therefore, from Lemma 4.3, the system of equations $\begin{cases} b^y-1=2^{x-1}\cdot l_1^{s_1} \\ b^y+1=2\cdot l_2^{s_2} \end{cases}$ is satisfied.

(1) From Proposition 4.2, We have the following results.

If $5^2-1=2^3\times 3^1$ is satisfied, then $l_2^{s_2}=(5^2+1)/2=13$. That is $l_2=13,\ s_2=1$.

If $3^4-1=2^4\times 5^1$ is satisfied, then $l_2^{s_2}=(3^4+1)/2=41$. That is $l_2=41,\ s_2=1$

If $17^2-1=2^5\times 3^2$ is satisfied, then $l_2^{s_2}=(17^2+1)/2=145$. Thus we have a contradiction to a prime number l_2.

If $7^2-1=2^4\times 3^1$ is satisfied, then $l_2^{s_2}=(7^2+1)/2=25$. That is $l_2=5,\ s_2=2$

(2) Let F be a Fermat prime number with $F\ge 17$. Then there exists a positive number n with $n\ge 2$ such that $F=2^{2^n}+1$ follows. And, if we put $b=2F-1=2^{2^n+1}+1$, from Lemma 2.3, the equation $b^2+1=2\cdot l_1^{s_1}$ has no solutions.

From Proposition 4.2, we may put $l_1 = F$ or $l_1 = M_0$. But the relation $l_1 = F$ does not occur for the reasons mentioned above. When $l_1 = M_0$ is satisfied, we put $b = 2M_0 + 1,\ r = \nu_2(M_0 + 1)$. Then we have $b^2 - 1 = 2^{r+2} \cdot M_0^{\ 1}$. Thus, if there exist an odd prime number p and a positive integer t such that $p^t = (b^2 + 1)/2$, there is only one solution $b^4 - 1 = 2^{r+3} \cdot M_0^{\ 1} \cdot p^t$. □

We consider solutions of equation $b^{2y} - 1 = 2^x \cdot l_1^{s_1} \cdot l_2^{s_2}$, where y is odd.

Lemma 4.5 $y \neq 1$ if and only if $b = 3$.

Proof Suppose $y = 1$. If $b = 3$, then $2^x \cdot l_1^{s_1} \cdot l_2^{s_2} = 3^2 - 1 = 8$. Thus we have a contradiction. Therefore $b \neq 3$ leads.
If $y \neq 1$ is satisfied, from Lemmma 2.2, $b^2 - 1 = 2^x$ follows. And, from $8 \mid b^2 - 1$, $x \geq 3$ is satisfied. Thus, from Catalan's Theorem, $b = 3$ leads. □

5. MAIN RESULTS

Let F be a Fermat prime number with $F \geq 17$. And let M, M_0 be a Mersenne prime number with $M \geq 31,\ M_0 \geq 7$.

Let b be odd, and let k be a positive integer. In this section, we consider the equation $(2^{x_{12}} - 1)/b^{y_1} = (b^{y_{12}} - 1)/2^{x_1} = k$.

We consider solutions of the equation $\dfrac{2^{x_{12}} - 1}{b^{y_1}} = \dfrac{b^{y^{12}} - 1}{2^{x_1}} = l_1^{s_1}$, where $y_{12} \neq 1$

Theorem 5.1 We have the following results:

(1) If b is an odd prime number, there are only two solutions $\dfrac{2^4 - 1}{5^1} = \dfrac{5^2 - 1}{2^3} = 3^1$ and $\dfrac{2^4 - 1}{3^1} = \dfrac{3^4 - 1}{2^4} = 5^1$.

(2) If $b = l_2^{s_2} \cdot l_3^{s_3}$ follows, There is no solution.

Proof First, we suppose that y_{12} is even.

If b is an odd prime number, from Proposition 4.2, the equation $b^{y_{12}} - 1 = 2^{x_1} \cdot l_1^{s_1}$ has only four solutions;

$$5^2 - 1 = 2^3 \cdot 3^1, \qquad 3^4 - 1 = 2^4 \cdot 5^1, \qquad 17^2 - 1 = 2^5 \cdot 3^2, \qquad 7^2 - 1 = 2^4 \cdot 3^1.$$

Thus we obtain either $b = 3$ or $l_1 = 3$. Therefore, from the other equation $2^{x_{12}} - 1 = b^{y_1} \cdot l_1^{s_1}$, $3 \mid 2^{x_{12}} - 1$ and so x_{12} is even. Hence, from Proposition 3.4, both $5^2 - 1 = 2^3 \cdot 3^1$ and $3^4 - 1 = 2^4 \cdot 5^1$ lead $2^4 - 1 = 5^1 \cdot 3^1$. Furthermore, $2^{x_{12}} - 1 = b^{y_1} \cdot l_1^{s_1}$ has no solution for $17^2 - 1 = 2^5 \cdot 3^2$ and $7^2 - 1 = 2^4 \cdot 3^1$.

If $b = l_2^{s_2} \cdot l_3^{s_3}$ follows, from Proposition 4.2, then either $l_1 = F$ or $l_1 = M_0$ is satisfied. Furthermore $s_1 = 1$ follows. Now, we put $b = 2F - 1$ or $b = 2M_0 + 1$ for each case. Then, from Lemma 2.4, b has an odd prime number p other than 3 and 3 as the factor. Thus $3 \mid 2^{x_{12}} - 1$ and so x_{12} is even. If $x_{12} \equiv 0 \pmod 4$ follows, from Proposition 3.8, the equation $2^{x_{12}} - 1 = b^{y_1} \cdot l_1^1$ has an unique solution $2^8 - 1 = 3^1 \cdot 5^1 \cdot 17^1$. This leads $b = 3^1 \cdot 5^1 = 15,\ l_1 = F = 17$. Thus we have a contradiction to $b = 2F - 1$. Therefore $x_{12} \equiv 2 \pmod 4$ is satisfied. Let $t \in \mathbb{N}$. Then, from Proposition 3.8, if the equation $2^{x_{12}} - 1 = b^{y_1} \cdot l_1^1$ has solutions, either $l_1 = F = (M+3)/2$, $b^{y_1} = 3^1 \cdot M^1$ or $l_1 = M_0 = M$, $b^{y_1} = 3^1 \cdot p^t = M + 2$ or $l_1 = M_0 = (M+3)/2$,

$b^{y_1} = 3^1 \cdot M^1$ follows. We remark that $b^{y_1} = 3^1 \cdot M^1$ or $b^{y_1} = 3^1 \cdot p^t$ leads $y_1 = 1$ each other.

If $l_1 = F$ then $3M = b = 2F - 1 = 2\{(M+3)/2\} - 1 = M + 2$, and so $M = 1$ follows. Thus we have a contradiction.

If $l_1 = M_0 = M$ then $M + 2 = b = 2M + 1$, and so $M = 1$ follows. Thus we have a contradiction.

If $l_1 = M_0 = (M+3)/2$ then $3M = b = 2M + 1$, and so $M = 1$ follows. Thus we have a contradiction.

Next, we suppose that y_{12} is an odd number with $y_{12} \neq 1$.

The equation $(b^{y_{12}} - 1)/2^{x_1} = l_1^{s_1}$ leads $2^{x_1} l_1^{s_1} = b^{y_{12}} - 1$. Thus, from Lemma 2.2, $b - 1 = 2^{x_1}$, and so $b = 2^{x_1} + 1$ follows. Therefore, the other equation $(2^{x_{12}} - 1)/b^{y_1} = l_1^{s_1}$, $2^{x_1} + 1 \mid 2^{x_{12}} - 1$ is obtained. Thus, from Lemma 2.5, both $x_{12} \equiv 0 \pmod 2$ and $x_1 \mid x_{12}/2$ follow.

If b is an odd prime number, from Proposition 3.4, the equation $2^{x_{12}} - 1 = b^{y_1} \cdot l_1^{s_1}$ has only two solutions;

$$2^4 - 1 = 3^1 \cdot 5^1, \qquad 2^6 - 1 = 3^2 \cdot 7^1.$$

If $b = 3,\ l_1^{s_1} = 5^1$ then $2^{x_1} = 3 - 1 = 2$ and so, $x_1 = 1$. Furthermore, from $(b^{y_{12}} - 1)/2^{x_1} = l_1^{s_1}$, $(3^{y_{12}} - 1)/2 = 5$ and so $3^{y_{12}} = 11$ follows. Thus we have a contradiction. On the other three cases $(b,\ l_1^{s_1}) = (5,\ 3^1),\ (3,\ 7^1),\ (7,\ 3^2)$, we have a contradiction similarly.

Suppose $b = l_2^{s_2} \cdot l_3^{s_3}$. If $x_{12} \equiv 0 \pmod 4$ follows, from Proposition 3.8, the equation $2^{x_{12}} - 1 = b^{y_1} \cdot l_1^{s_1}$ has an unique solution $2^8 - 1 = 3^1 \cdot 5^1 \cdot 17^1$. That is $b^{y_1} = 3^1 \cdot 5^1,\ 3^1 \cdot 17^1,\ 5^1 \cdot 17^1$. Hence $y_1 = 1$ and $b = 15,\ 51,\ 85$ follow. Thus we have a contradiction to $b = 2^{x_1} + 1$ for each other. Therefore $x_{12} \equiv 0 \pmod 4$ is satisfied. And, from Proposition 3.8, if the equation $2^{x_{12}} - 1 = b^{y_1} \cdot l_1^{s_1}$ has solutions then $b^{y_1} \cdot l_1^{s_1} = 3^1 \cdot M^1 \cdot p^t = 3^1 \cdot M^1 \cdot (M+2)/3$ follows at least. We remark that $y_1 = 1$ is obtained since $3^1 \,\|\, b^{y_1}$ or $M^1 \,\|\, b^{y_1}$ follows, where notation $a^n \,\|\, b$ means that $a^n \mid b$ is satisfied but $a^{n+1} \mid b$ is not satisfied.

If $b = 3M,\ l_1^{s_1} = (M+2)/3$, Then $(b^{y_{12}} - 1)/(b-1) = l_1^{s_1} = (M+2)/3 = (b+6)/9$, and so $-9 \equiv -6 \pmod b$ is satisfied. That is $3 \equiv 0 \pmod b$. Thus we have a contradiction. On the other two cases $(b,\ l_1^{s_1}) = (M+2,\ M^1),\ (M(M+2)/3,\ 3^1)$, we have a contradiction similarly. □

We consider solutions of equation $\dfrac{2^{2x} - 1}{b^{y_1}} = \dfrac{b^{y_{12}} - 1}{2^{x_1}} = l_1^{s_1} \cdot l_2^{s_2}$, where b is an odd prime number.

Theorem 5.2 The equation has no solutions.

Proof First, we suppose that x is even. From Proposition 3.8, The equation $2^{2x} - 1 = l_1^{s_1} \cdot l_2^{s_2} \cdot b^{y_1}$ has an unique solution $2^8 - 1 = 3^1 \cdot 5^1 \cdot 17^1$. Then we consider the other equation $b^{y_{12}} - 1 = 2^{x_1} \cdot l_1^{s_1} \cdot l_2^{s_2}$. If $b = 3$ follows, $3^{y_{12}} - 1 = 2^{x_1} \cdot 5^1 \cdot 17^1$ and so $5 \mid 3^{y_{12}} - 1$ is satisfied. That is $4 \mid y_{12}$. If $b = 5$ follows, $5^{y_{12}} - 1 = 2^{x_1} \cdot 3^1 \cdot 17^1$ and so $17 \mid 5^{y_{12}} - 1$ is satisfied. That is $16 \mid y_{12}$. If $b = 17$ follows, $17^{y_{12}} - 1 = 2^{x_1} \cdot 3^1 \cdot 5^1$ and so $5 \mid 17^{y_{12}} - 1$ is satisfied. That is $4 \mid y_{12}$. Thus $y_{12} \equiv 0 \pmod 4$ follows. Therefore, from Proposition 4.4, the equation $b^{y_{12}} - 1 = 2^{x_1} \cdot l_1^{s_1} \cdot l_2^{s_2}$ has no solutions for the above cases.

Next, we suppose that x is odd. From Proposition 3.8, if the equation $2^{2x} - 1 = l_1^{s_1} \cdot l_2^{s_2} \cdot b^{y_1}$ has solutions, there exist an odd prime number p and a positive integer t for M such that $2^{2m}-1=3^1\cdot M^1\cdot p^t=3^1\cdot M^1\cdot(M+2)/3$. Now, put $M=2^q-1$, where q is a suitable odd prime number with $q\ge 5$. Then we consider the other equation $b^{y_{12}}-1=2^{x_1}\cdot l_1^{s_1}\cdot l_2^{s_2}$.

In the case of $b=3$, the equation $3^{y_{12}}-1=2^{x_1}\cdot M^1\cdot(M+2)/3$ is satisfied. If $y_{12}\equiv 0 \pmod 4$ follows, from Proposition 4.4, there is no solution. If $y_{12}\equiv 2 \pmod 4$ follows, from Lemma 4.5, there exists an odd positive integer n with $n\ne 1$ such that $y_{12}=2n$. Then we have $x_1=\nu_2(3^{y_{12}}-1)=\nu_2(3^2-1)=3$. Furthermore, from $\gcd(3^n-1,\ 3^n+1)=2$ and Lemma 4.1, $2<3^n-1<3^n+1<2^2\cdot M\cdot(M+2)/3$ is satisfied. Hence, either $\begin{cases}3^n-1=2\cdot M\\ 3^n+1=4\cdot(M+2)/3\end{cases}$ or $\begin{cases}3^n-1=2\cdot(M+2)/3\\ 3^n+1=4\cdot M\end{cases}$ follows. Then the system of equation leads $(M,\ n)=(1,\ 1)$ for each case. Thus we have a contradiction. If y_{12} is odd, then $x_{12}=1$ is satisfied. Then we have $3^{y_{12}}-1=2^1\cdot(2^q-1)\cdot(2^q+1)/3$. That is $3^{y_{12}+1}-2^{2q+1}=1$. Thus we have a contradiction to Catalan's Theorem.

In the case of $b=M$, the equation $M^{y_{12}}-1=2^{x_1}\cdot 3^1\cdot(M+2)/3$ is satisfied. If $y_{12}\equiv 0 \pmod 4$ follows, from Proposition 4.4, there is no solution. If $y_{12}\equiv 2 \pmod 4$ follows, from Lemma 4.5, $y_{12}=2$ is satisfied. Therefore, $M^2-1=2^{x_1}(M+2)$, and so $(2^q-1)^2-1=2^{x_1}(2^q+1)$ is satisfied. That is $2^{q+1}(2^{q-1}-1)=2^{x_1}(2^q+1)$. Hence $x_1=q+1$ and $2^{q-1}-1=2^q+1$ follows. Since the inequality $2^{q-1}-1<2^q+1$ is satisfied clearly, we have a contradiction. If y_{12} is odd, we have $x_{12}=\nu_2(M-1)=\nu_2(2^q-2)=1$. Therefore, $M^{y_{12}}-1=2(M+2)$, and so $M^{y_{12}}=2M+5$ is satisfied. Thus we have a contradiction.

In the case of $b=p$, the equation $p^{y_{12}}-1=2^{x_1}\cdot 3^1\cdot M^1$ is satisfied. If $y_{12}\equiv 0 \pmod 4$ follows, from Proposition 4.4, there is no solution. If $y_{12}\equiv 2 \pmod 4$ follows, from Lemma 4.5, $y_{12}=2$ is satisfied. Therefore, $p^2=2^{x_1}\cdot 3M+1$, and so $\{(M+2)/3\}^2=(2^{x_1}\cdot 3M+1)^t$ is satisfied. Hence $4\equiv 9 \pmod M$, and so $0\equiv 5 \pmod M$ follows. Thus we have a contradiction. If $y_{12}=1$ follows, then $p=2^{x_1}\cdot 3M+1$, and so $(M+2)/3=(2^{x_1}\cdot 3M+1)^t$ is satisfied. Hence $2\equiv 3 \pmod M$, and so $0\equiv 1 \pmod M$ follows. Thus we have a contradiction. Suppose that y_{12} is an odd positive integer with $y_{12}\ne 1$. Then we remark that $2\mid p-1$ and $M\nmid p-1$ are satisfied. Therefore, from Lemma 2.2, either $p-1=2^{x_1}$ or $p-1=3\cdot 2^{x_1}$ follows. Therefore, from $p^t=(M+2)/3=(2^q+1)/3$, we have $2^q+1=3(k\cdot 2^{x_1}+1)^t$, where $k=1,\ 3$. Then, If $x_1\ge 2$ follows, $1\equiv 3 \pmod 4$, and so $0\equiv 2 \pmod 4$ follows. Thus we have a contradiction. If $x_1=1$ follows, $2^q+1=3^{t+1}$ or $2^q+1=3\cdot 7^t$ is satisfied. From Catalan's Theorem, $2^q+1=3^{t+1}$ has no solutions. And, from $7\nmid 2^q+1$, $2^q+1=3\cdot 7^t$ has no solutions too.

□

We consider solutions of the equation $\dfrac{2^{4x}-1}{b^{y_1}}=\dfrac{b^{y_{12}}-1}{2^{x_1}}=l_1^{s_1}\cdot l_2^{s_2}$, where $b=l_3^{s_3}\cdot l_4^{s_4}$.

Theorem 5.3 There is an unique solution $\dfrac{2^{12}-1}{91^1}=\dfrac{91^1-1}{2^1}=3^2\cdot 5^1$.

Proof From Proposition 3.9, the equation $2^{4x}-1=b^{y_1}\cdot l_1^{s_1}\cdot l_2^{s_2}$ has the following two solutions:

$$2^{12}-1=3^2\cdot 5^1\cdot 7^1\cdot 13^1,\qquad 2^{16}-1=3^1\cdot 5^1\cdot 17^1\cdot 257^1.$$

First, we consider the case of $2^{12}-1=3^2\cdot 5^1\cdot 7^1\cdot 13^1$. Then we remark that $y_1=1$ is satisfied.

Suppose $b=91$ .Then the other equation $91^{y_{12}}-1=2^{x_1}\cdot 3^2\cdot 5^1$ is satisfied. If y_{12} is even, $23\,|\,91^{y_{12}}-1$ follows. Thus we have a contradiction. If y_{12} is odd, we have $x_1=\nu_2(91-1)=1$. Therefore $91^{y_{12}}-1=2^1\cdot 3^2\cdot 5^1=90$, and so $y_{12}=1$ is obtained. Thus $\frac{2^{12}-1}{91^1}=\frac{91^1-1}{2^1}=3^2\cdot 5^1$ is a solution.

If $b=45$, the other equation $45^{y_{12}}-1=2^{x_1}\cdot 7^1\cdot 13^1$ follows. Thus we have a contradiction to $11\,|\,45^{y_{12}}-1$.

If $b=63$, the other equation $63^{y_{12}}-1=2^{x_1}\cdot 5^1\cdot 13^1$ follows. Thus we have a contradiction to $31\,|\,63^{y_{12}}-1$.

If $b=117$, the other equation $117^{y_{12}}-1=2^{x_1}\cdot 5^1\cdot 7^1$ follows. Thus we have a contradiction to $29\,|\,117^{y_{12}}-1$.

If $b=35$, the other equation $35^{y_{12}}-1=2^{x_1}\cdot 3^2\cdot 13^1$ follows. Thus we have a contradiction to $17\,|\,35^{y_{12}}-1$.

If $b=65$, the other equation $65^{y_{12}}-1=2^{x_1}\cdot 3^2\cdot 7^1$ follows. Then, from $3\,|\,65^{y_{12}}-1$, y_{12} is even. Thus we have a contradiction to $11\,|\,65^{y_{12}}-1$.

Next, we consider the case of $2^{16}-1=3^1\cdot 5^1\cdot 17^1\cdot 257^1$. Then we remark that $y_1=1$ is satisfied.

If $b=15$, the other equation $15^{y_{12}}-1=2^{x_1}\cdot 17^1\cdot 257^1$ follows. Thus we have a contradiction to $7\,|\,15^{y_{12}}-1$.

If $b=51$, the other equation $51^{y_{12}}-1=2^{x_1}\cdot 5^1\cdot 257^1$ follows. Thus we have a contradiction to $5^2\,|\,51^{y_{12}}-1$.

If $b=771$, the other equation $771^{y_{12}}-1=2^{x_1}\cdot 5^1\cdot 17^1$ follows. Thus we have a contradiction to $7\,|\,771^{y_{12}}-1$.

If $b=85$, the other equation $85^{y_{12}}-1=2^{x_1}\cdot 3^1\cdot 257^1$ follows. Thus we have a contradiction to $7\,|\,85^{y_{12}}-1$.

If $b=1285$, the other equation $1285^{y_{12}}-1=2^{x_1}\cdot 3^1\cdot 17^1$ follows. Thus we have a contradiction to $107\,|\,1285^{y_{12}}-1$.

If $b=4369$, the other equation $4369^{y_{12}}-1=2^{x_1}\cdot 3^1\cdot 5^1$ follows. Thus we have a contradiction to $7\,|\,4369^{y_{12}}-1$.

The proof is complete. □

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References

(1) P.Mihăilescu : "Primary Cyclotomic Units and a Proof of Catalan's Conjecture", J.Reinen Angew. Math. Vol. 572, pp.167-195 (2004).
(2) M.A.Bennett : "On Some Exponential Equation of S.S.Pillai", Canad J. Math. Vol. 53, No. 5, pp.897-922 (1980).
(3) N.Kobachi, Y.Motoda, and Y.Yamahata : "On Some Diophantine Equations (II)", Research Reports of NIT Kumamoto College Vol. 9, pp.83-90 (2017).